

TENSOR PRODUCTS FOR $SL(2, k)$

BY

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ABSTRACT. Let G be $SL(2, k)$ where k is a locally compact, nondiscrete, totally disconnected topological field whose residual characteristic is not 2, π_σ be a principal series representation of G , and $\pi \in \hat{G}$ be arbitrary. We determine the decomposition of $\pi_\sigma \otimes \pi$ into irreducibles by reducing this problem to decomposing the restriction of each $T \in \hat{G}$ to a minimal parabolic subgroup B of G and decomposing certain tensor products of irreducibles of B .

Let $G = SL(2, k)$ be the group of two by two matrices of determinant one over a locally compact, nondiscrete, totally disconnected topological field k whose residual characteristic is not 2. If

$$B = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} : a \in k^*, b \in k \right\}$$

and σ is a (unitary) character of B , i.e., a character on

$$C = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in k^* \right\},$$

then the family of representations $\{\pi_\sigma = \text{Ind}_B^G \sigma : \sigma \in \hat{C}\}$ is called the principal series of representations of G . The purpose of this paper is to determine the decomposition of $\pi_\sigma \otimes \pi$ into irreducibles where π_σ is a principal series representation and $\pi \in \hat{G}$ is arbitrary.

In §1 we summarize the results concerning the structure and representation theory of G that we shall use in the rest of the paper. Most of these results are well known. We refer to [2] for a more detailed treatment.

In §2, after using the Mackey theory to determine the representation theory of B , we decompose the restriction of each $\pi \in \hat{G}$ to B . This then, in combination with the Mackey-Anh reciprocity theorem, allows us to decompose $\text{Ind}_B^G T$ for all $T \in \hat{B}$.

In §3 we use Mackey's tensor product theorem to show how $\pi_\sigma \otimes \pi$ is

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determined from our knowledge of $\text{Ind}_B^G T$, $T \in \hat{B}$. In all cases it is shown that $\pi_\sigma \otimes \pi$ decomposes into two pieces, T_c and T_d , where T_c is a continuous direct sum with respect to Plancherel measure on \hat{G} of representations from the principal series of G and T_d is a discrete direct sum of representations from the discrete series of G and possibly the special representation of G . The multiplicity of a given representation occurring in $\pi_\sigma \otimes \pi$ is either 4, 2, or 1. These multiplicities vary not only as π_σ and π vary but also according to whether -1 is a square in k^* or not. For a specific choice of k , π_σ , and π , $\pi_\sigma \otimes \pi$ depends only upon the restrictions of π_σ and π to the center of G , $M = \{\pm e\}$.

The techniques developed in this paper apply also to the groups $\text{SL}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{R})$ and so we are able to decompose $\pi_\sigma \otimes \pi$ for these groups as well. We summarize the main results concerning this decomposition in §4 of this paper as well as indicate to what extent we have given new proofs of results previously obtained by G. Mackey in [4], L. Pukanszky in [6], F. Williams in [11], J. Repka in [7], and R. Martin in [5].

Throughout this paper we shall use freely Mackey's tensor product theorem, Mackey's subgroup theorem, and the Mackey-Anh reciprocity theorem. We refer to [5] or [1], [3], and [4] for the statements of these theorems.

1. Preliminaries. Let k be a locally compact, totally disconnected, nondiscrete topological field, k^+ the additive group, k^* the multiplicative group, \mathcal{O} the ring of integers in k , \mathfrak{P} the maximal ideal in \mathcal{O} , τ a generator of \mathfrak{P} , and $\mathcal{U} = \mathcal{O} - \mathfrak{P}$ the group of units in \mathcal{O} . \mathcal{O}/\mathfrak{P} is a finite field with q elements, q a prime power, and if we let ε denote a primitive $(q-1)$ st root of unity in \mathcal{U} , then $\{0, 1, \varepsilon, \dots, \varepsilon^{q-2}\}$ is a complete set of coset representatives for \mathcal{O}/\mathfrak{P} . *We shall assume throughout this paper that q is odd.* In this case, the set $E = \{1, \tau, \varepsilon, \varepsilon\tau\}$ is a complete set of coset representatives for $k^*/(k^*)^2$ and so, up to isomorphism, any quadratic extension of k can be expressed in the form $V_\alpha = k(\sqrt{\alpha})$, $\alpha \in E' = \{\tau, \varepsilon, \varepsilon\tau\}$. Let dx denote Haar measure on k^+ , normalized so that \mathcal{O} has measure 1. The equations $d(ax) = |a|dx$, $a \in k^*$, $|0| = 0$ determine a valuation on k for which $|\tau| = q^{-1}$, $\mathcal{O} = \{x: |x| \leq 1\}$, and $\mathfrak{P} = \{x: |x| < 1\}$.

For any nontrivial character χ on k^+ and $u \in k^+$, we let $\chi_u(x) = \chi(ux)$. The mapping $u \mapsto \chi_u$ is a topological isomorphism of k^+ onto \hat{k}^+ and so we may identify k^+ with \hat{k}^+ . For $f \in L^1(k^+)$, the (additive) Fourier transform is $(Ff)(u) = \int_k f(x)\chi(ux)dx$.

The set $\mathcal{U}_1 = \{x \in \mathcal{U}: |1-x| < 1\}$ is a compact subgroup of k^* and every element of k^* can be expressed as $x = \tau^n \varepsilon^i a$ where $n \in \mathbb{Z}$, $0 \leq i \leq q-2$, $a \in \mathcal{U}_1$. Thus $k^* \approx \mathbb{Z} \times \mathbb{Z}_{q-1} \times \mathcal{U}_1$ and $\hat{k}^* \approx T \times \mathbb{Z}_{q-1} \times \hat{\mathcal{U}}_1$ where T is isomorphic to the unit circle in the complex plane. It follows that every character $\sigma \in \hat{k}^*$ has the form $\sigma(x) = |x|^{\alpha} \sigma^*(x)$ where $\pi/\ln q \leq \alpha \leq \pi/\ln q$

and σ^* is a character on $\mathcal{U} = Z_{q-1} \times \mathcal{U}_1$. Thus we may view \hat{k}^* as the product of the usual Haar measure on the circle and a discrete measure.

If $\alpha \in E'$, we denote by k_{α^*} the set of elements in k^* of the form $x^2 - \alpha y^2$, $x, y \in k^*$. For each $\alpha \in E'$ we have $(k^*)^2 \subset k_{\alpha^*} \subset k^*$, $[k_{\alpha^*}: (k^*)^2] = [k^*: k_{\alpha^*}] = 2$, and $k_{\alpha^*}/(k^*)^2 \subset k^*/(k^*)^2$. We shall need more explicit knowledge of k_{α^*} later, so let $\mathcal{O}_{\alpha} = \{c^2\alpha: c \in k^*\}$ for $\alpha \in E$. Then $k^* = \mathcal{O}_1 \cup \mathcal{O}_{\tau} \cup \mathcal{O}_{\varepsilon} \cup \mathcal{O}_{\varepsilon\tau}$ and since $\varepsilon \in k_{\varepsilon^*}$, $k_{\varepsilon^*} = \mathcal{O}_1 \cup \mathcal{O}_{\varepsilon}$. If -1 is a square in k^* , then $-\tau \in \mathcal{O}_{\tau} \cap k_{\tau^*}$ and $k_{\tau^*} = \mathcal{O}_1 \cup \mathcal{O}_{\tau}$, $k_{\varepsilon\tau^*} = \mathcal{O}_1 \cup \mathcal{O}_{\varepsilon\tau}$. If -1 is not a square in k^* , then $-1 = x_1^2 - \varepsilon x_2^2$, $x_1, x_2 \in \mathcal{O}/\mathcal{P}$ implies $x_1 \neq 0 \neq x_2$ and so $-1 \in k_{\varepsilon^*}$. Thus $-1 \in \mathcal{O}_{\varepsilon}$, $-\tau \in \mathcal{O}_{\varepsilon\tau}$, $-\varepsilon\tau \in \mathcal{O}_{\tau}$, $k_{\tau^*} = \mathcal{O}_1 \cup \mathcal{O}_{\varepsilon\tau}$, and $k_{\varepsilon\tau^*} = \mathcal{O}_1 \cup \mathcal{O}_{\tau}$. We obtain 3 characters in \hat{k}^* given by

$$\text{sgn}_{\alpha}(x) = \begin{cases} 1, & x \in k_{\alpha^*}, \\ -1, & x \in k^* - k_{\alpha^*}, \alpha \in E'. \end{cases}$$

Let $G = SL(2, k)$ be the group of two by two matrices of determinant one over k and let

$$C = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in k^* \right\}, \quad V = \left\{ \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} : v \in k \right\}, \quad B = CV,$$

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in k \right\} \quad \text{and} \quad M = \{\pm e\}$$

be the center of G . Other than the unit representation, I , the following is a complete list of all the irreducible unitary representations in \hat{G} .

1. *The irreducible principal series* \mathcal{P} . The principal series of representations of G is the family $\{\pi_{\sigma} = \text{Ind}_B^G \sigma': \sigma \in \hat{C}\}$ where $\sigma'(cv) = \sigma(c)$. These representations are all irreducible except in the cases $\sigma_{\alpha} = \text{sgn}_{\alpha}$, $\alpha \in E'$, in which cases $\text{Ind}_B^G \sigma_{\alpha}$ decomposes as a direct sum of two irreducible representations which we shall denote by π_{σ}^{\pm} . It is known that $\pi_{\sigma} \simeq \pi_{\sigma'}$ iff $\sigma' = \sigma$ or σ^{-1} . Each π_{σ} may be realized on $L^2(k^+)$ by

$$\pi_{\sigma}(g)f(x) = \sigma(bx + d)|bx + d|^{-1}f\left(\frac{ax + c}{bx + d}\right) \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By using the Fourier transform F , we obtain the χ -realization for the representations of the principal series acting on $L^2(\hat{k}^+) = L^2(k^+)$. When $b = 0$ these representations have the following form

$$\pi_{\sigma}(g)f(x) = \sigma(d)|d|\chi(cdx)f(dx).$$

Letting M operate on \hat{C} by $e \cdot \sigma = \sigma$, $-e \cdot \sigma = \sigma^{-1}$ and letting $S = \hat{C}/M$, we then have that $\mathcal{P} = \{\pi_{\sigma}: \sigma \in S, (\sigma)^2 \neq 1\}$.

2. *The reducible principal series* \mathcal{R} . These are the irreducible representations arising as summands of the reducible principal series representations. So $\mathcal{R} = \{\pi_{\sigma_{\alpha}}^{\pm}: \alpha \in E'\}$. $\pi_{\sigma_{\alpha}}^+$ acts on the subspace H_{α}^+ of $L^2(k^+)$ of functions $f(u)$

for which $f(u) = 0$ when $\text{sgn}_\alpha u = -1$ while π_α^- acts on the subspace H_α^- of functions for which $f(u) = 0$ when $\text{sgn}_\alpha u = 1$.

3. *The complementary series* \mathcal{C} . The complementary series is the collection $\mathcal{C} = \{T_s: 0 < s < 1\}$. Each T_s may be realized in the space of functions $f(x)$ on k for which

$$(f, f) = \int_k |x|^{-s} |f(x)|^2 dx < \infty.$$

When $b = 0$, the action is given by

$$T_s(g)f(x) = |d|^{-s+1} \chi(dc x) f(d^2 x).$$

4. *The special representation* T_0 . The special representation acts in the space of functions $f(x)$ for which $(f, f) = \int_k |x|^{-1} |f(x)|^2 dx < \infty$. When $b = 0$, the action is given by

$$T_0(g)f(x) = \chi(dc x) f(d^2 x).$$

5. *The discrete series* \hat{G}_d . To each of the 3 quadratic extensions of k is associated a family of irreducible unitary representations called discrete series representations. If $V_\alpha = k(\sqrt{\alpha})$ is a quadratic extension of k , $C_\alpha = \{z = x + \sqrt{\alpha} y: x^2 - \alpha y^2 = 1\}$ is the unit circle in V_α , $\psi \in \hat{C}_\alpha$, and we also denote by ψ any extension of ψ to a character on V_α , then one obtains a (reducible) unitary representation $T(\psi, \alpha)$ acting in the space $L^2(k^+)$. When $b = 0$, the action is given by

$$T(\psi, \alpha)(g)f(x) = \text{sgn}_\alpha(d) \psi(d) |d| \chi(dc x) f(d^2 x).$$

$T(\psi, \alpha)$ splits into the direct sum of 2 inequivalent representations $T^+(\psi, \alpha)$ and $T^-(\psi, \alpha)$ where $T^+(\psi, \alpha)$ acts on the space H_α^+ [as in (2)] and $T^-(\psi, \alpha)$ acts on the space H_α^- . If ψ_α denotes the unique character on C_α of order two and $\psi \in \hat{C}_\alpha - \{\psi_\alpha\}$, then these representations are irreducible. However, contrary to the claims in [2] (see [9] and [10]), $T^+(\psi_\alpha, \alpha)$ and $T^-(\psi_\alpha, \alpha)$ each split into the direct sum of 2 inequivalent irreducibles, $T_i^+(\psi_\alpha, \alpha)$, $T_i^-(\psi_\alpha, \alpha)$, $i = 1, 2$. The set $\{T_i^\pm(\psi_\alpha, \alpha)\}$ is, up to unitary equivalence, independent of the choice of V_α and so we shall denote these representations by T_i^\pm , $i = 1, 2$. It is known that $T^+(\psi, \alpha) \simeq T^+(\psi', \alpha)$ (or $T^-(\psi, \alpha) \simeq T^-(\psi', \alpha)$) iff $\psi' = \psi$ or ψ^{-1} , $T^+(\psi, \alpha) \not\simeq T^-(\psi, \alpha)$ for any ψ, ψ' , and if $\alpha \neq \beta$, $\psi^2 \neq 1$, $(\psi')^2 \neq 1$, then $T^\pm(\psi, \alpha)$ and $T^\pm(\psi', \beta)$ are inequivalent. If for $\alpha \in E'$ we let M operate on \hat{C}_α by $e \cdot \psi = \psi$, $-e \cdot \psi = \psi^{-1}$ and let $S_\alpha = \hat{C}_\alpha / M$, we then have that $\hat{G}_d = \{T_i^\pm: i = 1, 2\} \cup \{T^\pm(\psi, \alpha): \psi \in S_\alpha, (\psi)^2 \neq 1, \alpha \in E'\}$.

So we now have that $\hat{G} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{C} \cup \{T_0\} \cup \hat{G}_d \cup \{I\}$. Plancherel measure μ_G on \hat{G} has support $\mathcal{P} \cup \mathcal{R} \cup \{T_0\} \cup \hat{G}_d$, is a discrete measure on \hat{G}_d , and is continuous on \mathcal{P} .

2. **Restrictions to B .** In this section we compute the restriction of each $\pi \in \hat{G}$ to the minimal parabolic subgroup B , $(\pi)_B$. In light of the Mackey-Anh reciprocity theorem, this will then allow us to determine $\text{Ind}_B^G T$ for each $T \in \hat{B}$. First we use the Mackey theory to determine the representation

The group

$$B = CV = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \right\}$$

is a semidirect product and, after identifying \hat{V} with V , the action of C on V is given by $c \cdot v = c^2 v$. Thus, in addition to the zero orbit Θ_0 , we have the 4 orbits Θ_α , $\alpha \in E$. The stability group corresponding to Θ_0 is C while the stability subgroups corresponding to the Θ_α , $\alpha \in E$, are all equal to the center of G , $M = \{\pm e\}$. Thus, corresponding to the orbit Θ_0 we get the (1-dimensional) irreducible representations of B lifted from the characters of C , i.e., the representations $\sigma(cv) = \sigma(c)$, $\sigma \in \hat{C}$, $cv \in CV$. We shall refer to these representations as the characters of B . Corresponding to the 4 nonzero orbits we obtain 8 infinite-dimensional irreducible representations $\rho_\alpha^\pm = \text{Ind}_{MV}^B(w^\pm \times \chi_\alpha)$, $\alpha \in E$, where $\hat{M} = \{w^\pm\}$ and $\chi_\alpha \in \hat{k}^+$ is $\chi_\alpha(u) = \chi(\alpha u)$. If μ_B denotes Plancherel measure on \hat{B} , then μ_B is a discrete measure whose support is $\{\rho_\alpha^\pm : \alpha \in E\}$.

By Mackey's subgroup theorem we have $(\rho_\alpha^\pm)_C \simeq \text{Ind}_M^C w^\pm$ and so by the Mackey-Anh reciprocity theorem, $(\rho_\alpha^\pm)_C \simeq \int_{\hat{C}^\pm} \sigma d\mu_C(\sigma)$ where $C^\pm = \{\sigma \in \hat{C} : (\sigma)_M = w^\pm\}$ and μ_C denotes Plancherel measure on \hat{C} . By arguing as in [5], we also have that MV and V are regularly related in B , a Borel cross-section for the $MV: V$ double cosets in B is $C/M = S$, and that for $s \in S$, $V \cap s^{-1}MV s = V$. Thus by Mackey's subgroup theorem

$$(\rho_\alpha^\pm)_V \simeq \int_S^\oplus \{n \mapsto \chi_\alpha(s \cdot n)\} d\gamma(s) \simeq \int_S^\oplus \chi_{s \cdot \alpha} d\gamma(s)$$

where γ denotes any admissible measure on S . Since any admissible measure on S will be nonatomic and have support S , we see that, after using the one-to-one correspondence between S and Θ_α to transfer γ to a nonatomic measure $\tilde{\gamma}$ on Θ_α whose support is all of Θ_α ,

$$(\rho_\alpha^\pm)_V \simeq \int_{\Theta_\alpha}^\oplus \chi_u d\tilde{\gamma}(u).$$

We now set about to determine $(\pi_\sigma)_B$ for each principal series representation π_σ . Recall that $\hat{C} \approx \hat{k}^*$ and any character $\sigma \in \hat{k}^*$ has the form $\sigma(x) = |x|^{is} \sigma^*(x)$ where $-\pi/\ln q \leq s \leq \pi/\ln q$ and $\sigma^* \in \hat{\mathcal{U}}$. We shall show that $(\pi_\sigma)_B \simeq \text{Ind}_C^B \sigma$ and so we must find $\text{Ind}_C^B \sigma$ for each $\sigma \in \hat{C}$. Since we have already found $(\delta)_C$ for almost all $\delta \in \hat{B}$, another application of the Mackey-Anh reciprocity theorem will give us $\text{Ind}_C^B \sigma$ for almost-all $\sigma \in \hat{C}$. The desired result will then follow from our knowledge of Plancherel measure on \hat{C} together with the following key theorem showing that inducing from C is independent of s (in fact inducing from C depends only on $(\sigma)_M$).

THEOREM 1. *Let $\sigma_1 = |\cdot|^{is_1} \sigma^*$ and $\sigma_2 = |\cdot|^{is_2} \sigma^*$. Then $\text{Ind}_C^B \sigma_1 \simeq \text{Ind}_C^B \sigma_2$.*

PROOF. Let $U_i = \text{Ind}_C^B \sigma_i$. By Mackey's subgroup theorem we know that

$$(\pi_{\sigma_i})_{CV} \simeq (\text{Ind}_B^G \sigma_i^{-1})_{CV} \simeq \text{Ind}_C^{CV} \sigma_i$$

and so U_i may be realized on $L^2(k^+)$ by

$$U_i \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} f(x) = \sigma_i(d) |d| \chi(dc x) f(d^2 x), \quad i = 1, 2.$$

Define $J: L^2(k^+) \rightarrow L^2(k^+)$ by $g(x) \mapsto |x|^{is_1/2} |x|^{-is_2/2} g(x)$. Then J is an isometry with the property that

$$\begin{aligned} \left(J U_1 \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} J^{-1} g \right)(x) &= \left(J U_1 \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} G \right)(x) \quad \text{where } J^{-1} g = G \\ &= |x|^{is_1/2} |x|^{-is_2/2} \sigma_1(d) |d| \chi(dc x) G(d^2 x) \\ &= |d|^{-is_1} |d|^{is_2} |d|^{is_1} \sigma^*(d) \chi(dc x) g(d^2 x) \\ &= \sigma_2(d) |d| \chi(dc x) g(d^2 x) \\ &= U_2 \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} g(x) \end{aligned}$$

and so $U_1 \simeq U_2$.

PROPOSITION 1. Let $\pi_{\sigma}, \sigma \in \hat{C}$, be a principal series representation. Then

$$(\pi_{\sigma})_B \simeq \begin{cases} \sum_{\alpha \in E} \oplus \rho_{\alpha}^+ & \text{if } (\sigma)_M = w^+, \\ \sum_{\alpha \in E} \oplus \rho_{\alpha}^- & \text{if } (\sigma)_M = w^-. \end{cases}$$

PROOF. It is easy to see that $\text{Ind}_{CN}^G \sigma \simeq \text{Ind}_{CV}^G \sigma^{-1} \simeq \text{Ind}_{CV}^G \sigma$ and so $\pi_{\sigma} \simeq \text{Ind}_{CN}^G \sigma$. Thus, since CVN has a complement of measure 0 in G , we have from Mackey's subgroup theorem that

$$(\pi_{\sigma})_B \simeq (\text{Ind}_{CN}^G \sigma)_B \simeq \text{Ind}_C^B \sigma.$$

Since $(\rho_{\alpha}^{\pm})_C \simeq \int_C^{\oplus} \sigma d\mu_C(\sigma)$ and $\{\rho_{\alpha}^{\pm}\}$ is μ_B -almost all of \hat{B} , the conclusion of this proposition is valid for almost every $\sigma \in \hat{C}$ by the Mackey-Anh reciprocity theorem. In light of Plancherel measure on \hat{C} and Theorem 1, the proposition holds for every $\sigma \in \hat{C}$.

We now use Proposition 1 to obtain explicit realizations of the ρ_{α}^{\pm} . This in turn will allow us to determine $(\pi)_{\sigma}$ for the remaining irreducible representations of G . So let $L^2(\theta_{\alpha})$ denote the subspace of functions in $L^2(k^+)$ which vanish off θ_{α} , $\alpha \in E$. Then

$$L^2(k^+) = \sum_{\alpha \in E} \oplus L^2(\theta_{\alpha})$$

and the $L^2(\theta_{\alpha})$ are closed invariant subspaces under $(\pi_{\sigma})_B$ for each $\sigma \in \hat{C}$.

Let π_σ^α denote $(\pi_\sigma)_B$ acting on $L^2(\mathcal{O}_\alpha)$. Then by repeating the argument of [2, p. 164] we see that each π_σ^α is irreducible and that by repeating the proof of [2, p. 197], $\pi_\sigma^\alpha \neq \pi_\sigma^\beta$ for $\alpha \neq \beta$. Now notice that for $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$\pi_\sigma^\alpha(g)f(u) = \chi(vu)f(u) = \chi_u(v)f(u), \quad u \in \mathcal{O}_\alpha,$$

and so $(\pi_\sigma^\alpha)_V$ is a direct integral with respect to the restriction of Haar measure on k^+ to \mathcal{O}_α of the characters χ_u , $u \in \mathcal{O}_\alpha$. Thus

$$\pi_\sigma^\alpha \simeq \begin{cases} \rho_\alpha^+ & \text{if } (\sigma)_M = w^+, \\ \rho_\alpha^- & \text{if } (\sigma)_M = w^-. \end{cases}$$

We now use the above realizations of the ρ_α^\pm to find the $(\pi)_B$ for $\pi \in \hat{G}_d$. Let $\psi \in \hat{C}_\alpha$ and $T(\psi, \alpha)$ be the corresponding unitary representation (as described in §1). Since $\sigma'(\cdot) = \text{sgn}_\alpha(\cdot)\psi(\cdot)$ is a multiplicative character on k^* , we have that $(T(\psi, \alpha))_B \simeq (\pi_{\sigma'})_B$. The latter, of course, will be either $\Sigma_{\alpha \in E} \oplus \rho_\alpha^+$ or $\Sigma_{\alpha \in E} \oplus \rho_\alpha^-$ according to whether $(\sigma')_M = w^+$ or not.

PROPOSITION 2. *If $\psi \in \hat{C}_\epsilon$, then*

$$\begin{aligned} (T^+(\psi, \epsilon))_B &\simeq \begin{cases} \rho_1^+ \oplus \rho_\epsilon^+ & \text{if } (\psi)_M = w^+, \\ \rho_1^- \oplus \rho_\epsilon^- & \text{if } (\psi)_M = w^-, \end{cases} \\ (T^-(\psi, \epsilon))_B &\simeq \begin{cases} \rho_\tau^+ \oplus \rho_{\epsilon\tau}^+ & \text{if } (\psi)_M = w^+, \\ \rho_\tau^- \oplus \rho_{\epsilon\tau}^- & \text{if } (\psi)_M = w^-. \end{cases} \end{aligned}$$

PROOF. This follows immediately by looking at the realizations of the $T^\pm(\psi, \epsilon)$ given in §1, the realizations of the ρ_α^\pm given above, and the facts that $k_{\epsilon^*} = \mathcal{O}_1 \cup \mathcal{O}_\epsilon$ and $(\text{sgn}_\epsilon)_M = w^+$.

PROPOSITION 3. *Let $\{\alpha, \alpha'\} = \{\tau, \epsilon\tau\}$ and $\psi \in \hat{C}_\alpha$. Then*

(i) *if $-1 \in (k^*)^2$, then $k_{\alpha^*} = \mathcal{O}_1 \cup \mathcal{O}_\alpha$ and*

$$\begin{aligned} (T^+(\psi, \alpha))_B &\simeq \begin{cases} \rho_1^+ \oplus \rho_\alpha^+ & \text{if } (\psi)_M = w^+, \\ \rho_1^- \oplus \rho_\alpha^- & \text{if } (\psi)_M = w^-, \end{cases} \\ (T^-(\psi, \alpha))_B &\simeq \begin{cases} \rho_\epsilon^+ \oplus \rho_{\alpha'}^+ & \text{if } (\psi)_M = w^+, \\ \rho_\epsilon^- \oplus \rho_{\alpha'}^- & \text{if } (\psi)_M = w^-, \end{cases} \end{aligned}$$

while

(ii) *if $-1 \notin (k^*)^2$, then $k_{\alpha^*} = \mathcal{O}_1 \cup \mathcal{O}_{\alpha'}$ and*

$$\begin{aligned} (T^+(\psi, \alpha))_B &\simeq \begin{cases} \rho_1^- \oplus \rho_{\alpha'}^- & \text{if } (\psi)_M = w^+, \\ \rho_1^+ \oplus \rho_{\alpha'}^+ & \text{if } (\psi)_M = w^-, \end{cases} \\ (T^-(\psi, \alpha))_B &\simeq \begin{cases} \rho_\epsilon^- \oplus \rho_\alpha^- & \text{if } (\psi)_M = w^+, \\ \rho_\epsilon^+ \oplus \rho_\alpha^+ & \text{if } (\psi)_M = w^-. \end{cases} \end{aligned}$$

PROOF. This is again immediate once one sees that for $-1 \in (k^*)^2$, $(\text{sgn}_\alpha)_M = w^+$ while for $-1 \notin (k^*)^2$, $(\text{sgn}_\alpha)_M = w^-$.

PROPOSITION 4. $(\pi_{\alpha^\pm})_B \simeq (T^\pm(\psi, \alpha))_B$ for $(\psi)_M = w^+$ and $\alpha \in E'$.

PROOF. This follows easily from the facts that $(\sigma_e)_M = w^+$ while for $\alpha \in \{\tau, \varepsilon\tau\}$, $(\sigma_\alpha)_M = w^+$ if $-1 \in (k^*)^2$ and $(\sigma_\alpha)_M = w^-$ if $-1 \notin (k^*)^2$.

PROPOSITION 5. (i) If $-1 \in (k^*)^2$, then

$$(T_1^+ \oplus T_2^+ \oplus T_1^- \oplus T_2^-)_B \simeq \rho_1^- \oplus \rho_e^- \oplus \rho_\tau^- \oplus \rho_{\varepsilon\tau}^-.$$

(ii) If $-1 \notin (k^*)^2$, then

$$(T_1^+ \oplus T_2^+ \oplus T_1^- \oplus T_2^-)_B \simeq \rho_1^+ \oplus \rho_e^+ \oplus \rho_\tau^+ \oplus \rho_{\varepsilon\tau}^+.$$

PROOF. Let $\alpha = \tau$ and ψ_τ be the unique character of order 2 on C_τ . Then $(\psi_\tau)_M = w^-$ while

$$(\text{sgn}_\tau)_M = \begin{cases} w^+ & \text{if } -1 \in (k^*)^2, \\ w^- & \text{if } -1 \notin (k^*)^2, \end{cases}$$

and so the proposition follows.

Proposition 5 shows that the four discrete series representations T_i^\pm are completely determined by their restrictions to B . So, rather than turn to realizations of these representations (which we have not given) to find each $(T_i^\pm)_B$, we simply adopt the following conventions. If $-1 \in (k^*)^2$, we let T_1^+ , T_2^+ , T_1^- , T_2^- denote the discrete series representations whose restrictions to B are ρ_1^- , ρ_e^- , ρ_τ^- , $\rho_{\varepsilon\tau}^-$ respectively. If $-1 \notin (k^*)^2$, we let T_1^+ , T_2^+ , T_1^- , T_2^- denote the discrete series representations whose restrictions to B are ρ_1^+ , ρ_e^+ , ρ_τ^+ , and $\rho_{\varepsilon\tau}^+$ respectively. From now on we assume that the T_i^\pm are the above representations.

PROPOSITION 6. If T_0 denotes the special representation, then

$$(T_0)_B \simeq \sum_{\alpha \in E} \oplus \rho_\alpha^+.$$

PROOF. $(T_0)_B$ acts on $H = L^2(k^+, dx/|x|)$ by

$$T_0 \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} f(u) = \chi(dc) f(d^2u).$$

If we define $I: L^2(k^+, dx) \rightarrow H$ by $f(x) \mapsto |x|^{1/2} f(x)$, then I is an isometry with the additional property that

$$\begin{aligned} \left(I \pi_\sigma \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} I^{-1} f \right)(x) &= |x|^{1/2} \sigma(d) |d| \chi(dc) F(d^2x), \quad \text{where } I^{-1} f = F \\ &= |x|^{1/2} \sigma(d) |d| \chi(dc) |d|^{-1} |x|^{-1/2} f(d^2x) \\ &= \sigma(d) \chi(dc) f(d^2x). \end{aligned}$$

Thus we see that

$$(T_0)_B \simeq (\pi_1)_B \simeq \sum_{\alpha \in E} \oplus \rho_\alpha^+.$$

PROPOSITION 7. *If T_s is in the complementary series of G , then for each $0 < s < 1$, $(T_s)_B \simeq \sum_{\alpha \in E} \oplus \rho_\alpha^+$.*

PROOF. Same as the proof of Proposition 6 using $I: f(x) \mapsto |x|^{s/2}f(x)$.

The following theorem is now an immediate consequence of Propositions 1–7 and the Mackey-Anh reciprocity theorem.

THEOREM 2. *Let*

$$\mathcal{P}^\pm = \{\pi_\sigma \in \mathcal{P}: (\sigma)_M = w^\pm\}, \quad T_c^\pm = \int_{\mathcal{P}^\pm}^\oplus \pi d\mu_G(\pi),$$

$$T_\alpha^+ = T_0 \oplus \sum \oplus \{\pi \in \hat{G}_d: \rho_\alpha^+ \leq (\pi)_B\},$$

and

$$T_\alpha^- = \sum \oplus \{\pi \in \hat{G}_d: \rho_\alpha^- \leq (\pi)_B\}.$$

Then $\text{Ind}_B^G \rho_\alpha^\pm \simeq T_c^\pm \oplus T_\alpha^\pm$.

REMARKS. (1) We now know $\text{Ind}_B^G T$ for each $T \in \hat{B}$ since we have already noted that when one induces the characters of B up to G one gets the principal series of G .

(2) We have taken the liberty of not including the reducible principal series representations in T_c^\pm since they constitute a set of μ_G -measure 0 in \hat{G} .

3. Decomposing $\pi_\sigma \otimes \pi$. Let π_σ be a principal series representation and $\pi \in \hat{G}$. Then

$$\pi_\sigma \otimes \pi \simeq \text{Ind}_B^G \sigma \otimes \text{Ind}_G^G \pi$$

and we may apply Mackey's tensor product theorem to conclude

$$\pi_\sigma \otimes \pi \simeq \text{Ind}_B^G \{\sigma \otimes (\pi)_B\}.$$

LEMMA. *Let σ be a character on B and ρ_α^\pm be one of the 8 infinite-dimensional representations of B . Then*

$$\sigma \otimes \rho_\alpha^\pm \simeq \begin{cases} \rho_\alpha^\pm & \text{if } (\sigma)_M = w^+, \\ \rho_\alpha^\mp & \text{if } (\sigma)_M = w^-. \end{cases}$$

PROOF. By another application of Mackey's tensor product theorem we have

$$\sigma \otimes \rho_\alpha^\pm \simeq \text{Ind}_B^B \sigma \otimes \text{Ind}_{M \backslash V}^B (w^\pm \times \chi_\alpha) \simeq \text{Ind}_{M \backslash V}^B \{(\sigma)_M w^\pm \times \chi_\alpha\}$$

and the lemma follows.

We shall use the following notation throughout the remainder of this

section:

$$\mathcal{O}^\pm = \{\pi_\sigma \in \mathcal{O}: (\sigma)_M = w^\pm\}, \quad T_c^\pm = \int_{\mathcal{O}^\pm}^\oplus \pi d\mu_G(\pi),$$

$$S_\alpha^\pm = \{\psi \in S_\alpha: (\psi)^2 \neq 1 \text{ and } (\psi)_M = w^\pm\},$$

$$D^\pm(T^+, \alpha) = \sum_{\psi \in S_\alpha^\pm} \oplus T^+(\psi, \alpha),$$

and

$$D^\pm(T^-, \alpha) = \sum_{\psi \in S_\alpha^\pm} \oplus T^-(\psi, \alpha), \quad \alpha \in E'.$$

We shall state the following theorems for the case $-1 \in (k^*)^2$ and later indicate the changes needed for the case $-1 \notin (k^*)^2$.

THEOREM 3. *Let π_σ and π_γ be two principal series representations of G . Then if $-1 \in (k^*)^2$, we have*

$$\pi_\sigma \otimes \pi_\gamma \simeq \begin{cases} 4T_c^+ \oplus 4T_0 \oplus 2 \sum_{\alpha \in E'} \oplus \{D^+(T^+, \alpha) \oplus D^+(T^-, \alpha)\} & \text{if } (\sigma\gamma)_M = w^+, \\ 4T_c^- \oplus 2 \sum_{\alpha \in E'} \oplus \{D^-(T^+, \alpha) \oplus D^-(T^-, \alpha)\} & \\ \oplus \sum_{i=1,2} \oplus (T_i^+ \oplus T_i^-) & \text{if } (\sigma\gamma)_M = w^-. \end{cases}$$

PROOF. For $\pi = \pi_\gamma$ we have

$$\sigma \otimes (\pi)_B \simeq \begin{cases} \sum_{\alpha \in E} \oplus \rho_\alpha^+ & \text{if } (\sigma\gamma)_M = w^+, \\ \sum_{\alpha \in E} \oplus \rho_\alpha^- & \text{if } (\sigma\gamma)_M = w^- \end{cases}$$

and

$$\pi_\sigma \otimes \pi \simeq \text{Ind}_B^G \{\sigma \otimes (\pi)_B\} \simeq \sum_{\alpha \in E} \oplus \text{Ind}_B^G \rho_\alpha^\pm.$$

So Theorem 3 follows from Theorem 2.

THEOREM 4. *Let π be the special representation T_0 or a complementary series representation T_s . Then $\pi_\sigma \otimes \pi \simeq \pi_\sigma \otimes \pi_1$ where π_1 is the principal series representation corresponding to the trivial representation $1 \in \hat{C}$.*

PROOF. Immediate since $(\pi)_B \simeq (\pi_1)_B$.

THEOREM 5. *Let $\pi = T^\pm(\psi, \epsilon)$ with $\psi \in \hat{C}_\epsilon$ and $-1 \in (k^*)^2$. Then*

$$\pi_\sigma \otimes \pi \simeq \begin{cases} 2T_c^+ \oplus 2T_0 \oplus 2D^+(T^\pm, \epsilon) \\ \oplus \sum_{\tau, \epsilon\tau} \{D^+(T^+, \alpha) \oplus D^+(T^-, \alpha)\} & \text{if } (\sigma)_M(\psi)_M = w^+, \\ 2T_c^- \oplus 2D^-(T^\pm, \epsilon) \oplus T_1^\pm \oplus T_2^\pm \\ \oplus \sum_{\tau, \epsilon\tau} \{D^-(T^+, \alpha) \oplus D^-(T^-, \alpha)\} & \text{if } (\sigma)_M(\psi)_M = w^-. \end{cases}$$

PROOF. Theorem 5 follows directly from the facts that if $\pi = T^+(\psi, \epsilon)$, then $\pi_\sigma \otimes \pi$ is equivalent to either $\text{Ind}_B^G\{\rho_1^+ \oplus \rho_\epsilon^+\}$ or $\text{Ind}_B^G\{\rho_1^- \oplus \rho_\epsilon^-\}$ according to whether $(\sigma)_M(\psi)_M = w^+$ or w^- while if $\pi = T^-(\psi, \epsilon)$, $\pi_\sigma \otimes \pi$ is equivalent to either $\text{Ind}_B^G\rho_\tau^+ \oplus \rho_{\epsilon\tau}^+$ or $\text{Ind}_B^G\rho_\tau^- \oplus \rho_{\epsilon\tau}^-$ according to whether $(\sigma)_M(\psi)_M = w^+$ or w^- .

The following theorems are proven similarly.

THEOREM 6. Let $\pi = T^\pm(\psi, \alpha)$ for $\psi \in \hat{C}_\alpha$, $\alpha \in \{\tau, \epsilon\tau\}$, and let $-1 \in (k^*)^2$. Then

$$\pi_\sigma \otimes \pi \simeq \begin{cases} 2T_c^+ \oplus 2T_0 \oplus 2D^+(T^\pm, \alpha) \\ \oplus \sum_{\theta=\epsilon, \alpha'} \{D^+(T^+, \theta) \oplus D^+(T^-, \theta)\} \\ \text{if } (\sigma)_M(\psi)_M = w^+, \\ 2T_c^- \oplus 2D^-(T^\pm, \alpha) \oplus \sum_{\theta=\epsilon, \alpha'} \{D^-(T^+, \theta) \oplus D^-(T^-, \theta)\} \\ \oplus J & \text{if } (\sigma)_M(\psi)_M = w^-, \end{cases}$$

where J is $T_1^+ \oplus T_1^-$ for $T^+(\psi, \tau)$, $T_2^+ \oplus T_2^-$ for $T^-(\psi, \tau)$, $T_1^+ \oplus T_2^-$ for $T^+(\psi, \epsilon\tau)$, and $T_2^+ \oplus T_1^-$ for $T^-(\psi, \epsilon\tau)$.

THEOREM 7. Let $\pi = T_1^+$ and $-1 \in (k^*)^2$. Then

$$\pi_\sigma \otimes \pi \simeq \begin{cases} T_c^- \oplus D^-(T^+, \epsilon) \oplus D^-(T^+, \tau) \oplus D^-(T^+, \epsilon\tau) \oplus T_1^+ \\ \text{if } (\sigma)_M = w^+, \\ T_c^+ \oplus T_0 \oplus D^+(T^+, \epsilon) \oplus D^+(T^+, \tau) \oplus D^+(T^+, \epsilon\tau) \\ \text{if } (\sigma)_M = w^-. \end{cases}$$

THEOREM 8. Let $\pi = T_2^+$ and $-1 \in (k^*)^2$. Then

$$\pi_\sigma \otimes \pi \simeq \begin{cases} T_c^- \oplus T_2^+ \oplus D^-(T^+, \epsilon) \oplus D^-(T^-, \tau) \oplus D^-(T^-, \epsilon\tau) \\ \text{if } (\sigma)_M = w^+, \\ T_c^+ \oplus T_0 \oplus D^+(T^+, \epsilon) \oplus D^+(T^-, \tau) \oplus D^+(T^-, \epsilon\tau) \\ \text{if } (\sigma)_M = w^-. \end{cases}$$

THEOREM 9. Let $\pi = T_1^-$ and $-1 \in (k^*)^2$. Then

$$\pi_\sigma \otimes \pi \simeq \begin{cases} T_c^- \oplus T_1^- \oplus D^-(T^-, \varepsilon) \oplus D^-(T^+, \tau) \oplus D^-(T^-, \varepsilon\tau) & \text{if } (\sigma)_M = w^+, \\ T_c^+ \oplus T_0 \oplus D^+(T^-, \varepsilon) \oplus D^+(T^+, \tau) \oplus D^+(T^-, \varepsilon\tau) & \text{if } (\sigma)_M = w^-. \end{cases}$$

THEOREM 10. Let $\pi = T_2^-$ and $-1 \in (k^*)^2$. Then

$$\pi_\sigma \otimes \pi \simeq \begin{cases} T_c^- \oplus T_2^- \oplus D^-(T^-, \varepsilon) \oplus D^-(T^-, \tau) \oplus D^-(T^+, \varepsilon\tau) & \text{if } (\sigma)_M = w^+, \\ T_c^+ \oplus T_0 \oplus D^+(T^-, \varepsilon) \oplus D^+(T^-, \tau) \oplus D^+(T^+, \varepsilon\tau) & \text{if } (\sigma)_M = w^-. \end{cases}$$

THEOREM 11. $(\pi_{\alpha^\pm})_B \simeq (T^\pm(\psi, \alpha))_B$ where $(\psi)_M = w^\mp$.

REMARK. For $-1 \notin (k^*)^2$, the following changes must be made in the above theorems: as noted previously, we take T_1^+ , T_2^+ , T_1^- , T_2^- , to be those discrete series representations whose restrictions to B are ρ_1^+ , ρ_e^+ , $\rho_{e\tau}^+$, ρ_τ^+ respectively; for $\alpha = \tau, \varepsilon\tau$ we define

$$S_\alpha^\pm = \{\psi \in S_\alpha: (\psi)^2 \neq 1, (\psi)_M = w^\mp\};$$

place the representations T_i^\pm (whenever they occur) in the same decomposition as T_c^+ (instead of T_c^-); and interchange the phrases "if $(\sigma)_M(\psi)_M = w^+$ " and "if $(\sigma)_M(\psi)_M = w^-$ " in Theorems 6, 7, 8, 9, and 10.

4. Some comments on $\text{SL}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{R})$. The techniques developed in this paper apply also to the groups $\text{SL}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{R})$. As the proofs are easy modifications of those in §§2 and 3, we shall simply summarize the major results.

A. $G = \text{SL}(2, \mathbb{C})$. In this case $C = \mathbb{C}^*$ acts on $V = \mathbb{C}[c \cdot v = c^2v]$ in one nonzero orbit. Since the stability subgroup of a point, say 1, in the nonzero orbit is also $M = \{\pm e\}$, we see that there are just 2 infinite-dimensional representations in B , viz., $\rho^\pm = \text{Ind}_{M_V}^B(w^\pm \times \chi_1)$. $\hat{C} = \{\sigma(n, s): n \in \mathbb{Z}, s \in \mathbb{R}\}$ where $\sigma(n, s)(z) = (z/|z|)^n |z|^{is}$ and there are no characters of order 2. If n is even, then $(\sigma(n, s))_M = w^+$ while if n is odd, $(\sigma(n, s))_M = w^-$. If $\pi_{\sigma(n, s)} = \pi(n, s)$ is a principal series representation, then

$$(\pi(n, s))_B \simeq \begin{cases} \rho^+ & \text{if } n \text{ is even,} \\ \rho^- & \text{if } n \text{ is odd.} \end{cases}$$

If T_s is a complementary series representation, then $(T_s)_B \simeq (\pi(0, 0))_B$. So $\hat{G} = \mathcal{P} \cup \mathcal{C} \cup \{I\}$ where $\mathcal{P} = \{\pi(n, s): n = 0 \text{ and } s > 0 \text{ or } n \in \mathbb{N} \text{ and } s \in \mathbb{R}\}$ and $\mathcal{C} = \{T_s: 0 < s < 1\}$. Letting $\mathcal{P}^+ = \{\pi(n, s) \in \mathcal{P}: n \text{ is even}\}$,

$\mathcal{P}^- = \{\pi(n, s) \in \mathcal{P} : n \text{ is odd}\}$, and $T_c^\pm = \int_{\mathcal{P}^\pm} \pi d\mu_G(\pi)$, we have:

THEOREM. *Let $G = SL(2, \mathbb{C})$, $\pi(n, s)$ and $\pi(m, t)$ be two principal series representations of G , and T_u , $0 < u < 1$, be a complementary series representation of G . Then*

$$(i) \quad \pi(n, s) \otimes \pi(m, t) \simeq \begin{cases} T_c^+ & \text{if } n, m \text{ are both even or both odd,} \\ T_c^- & \text{otherwise,} \end{cases}$$

$$(ii) \quad \pi(n, s) \otimes T_u \simeq \pi(n, s) \otimes \pi(0, 0).$$

Case (i) was first proven by Mackey [4] and later by Williams [11] and Martin [5].

B. $G = SL(2, \mathbb{R})$. In this case $C = \mathbb{R}^*$ acts on $V = \mathbb{R}$ in two nonzero orbits and so there are 4 infinite-dimensional representations in \hat{B} , $\rho_{\pm 1}^\pm = \text{Ind}_{MV}^B(w^\pm \times \chi_{\pm 1})$. $\hat{C} = \{\sigma(h, s) : h = 0, 1, s \in \mathbb{R}\}$ where $\sigma(h, s)(x) = (x/|x|)^h |x|^s$. The character $\sigma(1, 0)$ is the only character of order 2 and $(\sigma(0, s))_M = w^+$ while $(\sigma(1, s))_M = w^-$. If $\pi(h, s)$ is a principal series representation of G , then

$$(\pi(h, s))_B \simeq \begin{cases} \rho_1^+ \oplus \rho_{-1}^+ & \text{if } h = 0, \\ \rho_1^- \oplus \rho_{-1}^- & \text{if } h = 1. \end{cases}$$

If T_u , $0 < u < 1$, is a complementary series representation, then $(T_u)_B \simeq (\pi(0, 0))_B$. If D_n , $n \in \{\pm 1, \pm 3/2, \dots\}$, is a discrete series representation of G , then $(D_n)_B$ is equivalent to ρ_1^+ for $n = 1, 2, \dots$, ρ_{-1}^+ for $n = -1, -2, \dots$, ρ_1^- for $n = 3/2, \dots$, and ρ_{-1}^- for $n = -3/2, \dots$. Letting $\pi(1, 0) = D_{1/2} \oplus D_{-1/2}$, we have $(D_{1/2})_B \simeq \rho_1^-$ while $(D_{-1/2})_B \simeq \rho_{-1}^-$. So $\hat{G} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{C} \cup \hat{G}_d \cup \{I\}$ where $\mathcal{P} = \{\pi(h, s) : s > 0 \text{ if } h = 0 \text{ and } s > 0 \text{ if } h = 1\}$, $\mathcal{R} = \{D_{1/2}, D_{-1/2}\}$, $\mathcal{C} = \{T_u : 0 < u < 1\}$, and $\hat{G}_d = \{D_n : n = \pm 1, \pm 3/2, \dots\}$. Letting $\mathcal{P}^+ = \{\pi(0, s) : s \geq 0\}$, $\mathcal{P}^- = \{\pi(1, s) : s > 0\}$, $T_c^\pm = \int_{\mathcal{P}^\pm} \pi d\mu_G(\pi)$, $I^+ = \{1, 2, \dots\}$, $I^- = \{-1, -2, \dots\}$, $J^+ = \{3/2, \dots\}$ and $J^- = \{-3/2, \dots\}$ we have

THEOREM. *Let G be $SL(2, \mathbb{R})$, $\pi(h, s)$, $\pi(m, t)$ be two principal series representations of G and T_u be a complementary series representation of G . Then*

$$(i) \quad \pi(h, s) \otimes \pi(m, t) \simeq \begin{cases} 2T_c^+ \oplus \sum_{I^+ \cup I^-} \oplus D_n & \text{if } h \text{ and } m \text{ are both 0 or both 1,} \\ 2T_c^- \oplus \sum_{J^+ \cup J^-} \oplus D_n & \text{otherwise,} \end{cases}$$

$$(ii) \quad \pi(h, s) \otimes T_u \simeq \pi(h, s) \otimes \pi(0, 0),$$

$$\begin{aligned}
 \text{(iii)} \quad \pi(0, s) \otimes D_n &\simeq \begin{cases} T_c^+ \oplus \sum_{I^+} \oplus D_i & \text{if } n \in I^+, \\ T_c^+ \oplus \sum_{I^-} \oplus D_i & \text{if } n \in I^-, \\ T_c^- \oplus \sum_{J^+} \oplus D_i & \text{if } n \in J^+ \cup \{1/2\}, \\ T_c^- \oplus \sum_{J^-} \oplus D_i & \text{if } n \in J^- \cup \{-1/2\}, \end{cases} \\
 \text{(iv)} \quad \pi(1, s) \otimes D_n &\simeq \begin{cases} T_c^- \oplus \sum_{J^+} \oplus D_i & \text{if } n \in I^+, \\ T_c^- \oplus \sum_{J^-} \oplus D_i & \text{if } n \in I^-, \\ T_c^+ \oplus \sum_{I^+} \oplus D_i & \text{if } n \in J^+ \cup \{1/2\}, \\ T_c^+ \oplus \sum_{I^-} \oplus D_i & \text{if } n \in J^- \cup \{-1/2\}. \end{cases}
 \end{aligned}$$

One may find the above decompositions, as well as all the other possible cases for the group $SL(2, \mathbf{R})$, in Repka [7]. The cases (i) and (ii) were first obtained by Pukanszky in [6]. A proof of (i) may also be found in [5].

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